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The relative efficiency of split–plot × split–block designs and split–block–plot designs

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SUMMARY

The paper deals with the examination of the relative efficiency of two mixed designs with reference to the accuracy of comparisons estimation of treatment parameters. One of the designs is called a split-plot × split-block design whereas the second one a split-block-plot design (strip-split-plot design). Contexts in which the designs are equally efficient and when one of them is more efficient than the other one are presented.

Key words: Mixed designs, Relative efficiency, Split–block–plot design, Split–plot × split–block design, Strip–split–plot design

1. Introduction

Experimental designs used in agricultural research for three-or-more factor experiments are certain extensions of either the split-plot or the split-block design known from literature. Considered here two mixed designs are combinations of the designs mentioned above. One of the mixed designs is called the split-block-plot (SBP) design (e.g. Ambroży and Mejza, 2002, Mejza and Ambroży, 2003). Another term of it is the strip-split-plot design (e.g. Gomez and Gomez, 1984). The SBP design is an extension of the split-block design in which an intersection plot is divided into subplots to accommodate levels of the third factor. Therefore, the third factor is in the split-plot design in a relation to row and column treatments (i.e. combinations of levels of two first factors). In field and glasshouse trials the SBP designs are commonly used in practice. The second mixed design presented here differs only from the SBP design in an arrangement of the levels of the third factor. Certain treatments such as types of cultivation, application of irrigation water etc., may be necessary to arrange them in strips (rows or columns) across each block. The columns (or the rows) of the split-block design should be split into smaller strips to accommodate the third factor. In this way, the third factor will be in the split-plot design in a relation to the column (row) treatments. As a result, the mixed design of the crossed split-plot and split-block designs is called the split-plot $\times$split-block (SPSB) design (e.g. LeClerg et al., 1962, Ambroży and Mejza, 2004).

The aim of this paper is to compare an effectiveness of the two mixed designs under mixed linear models in the context of an estimation of certain groups of contrasts. Those considerations refer to incomplete designs with orthogonal block structure as well as complete designs. A design is treated as incomplete if the numbers of special experimental units, i.e. rows, columns, plots in blocks to accommodate the factors are smaller than the numbers of their levels. We can also say, that the design is non-orthogonal with respect to those types of the treatments.

2. Assumptions and notations

Let us consider a three-factor experiment in which the first factor, say A, has s levels A_1, A_2, ..., A_s, the second factor, say B, has t levels B_1, B_2, ..., B_t and the third factor, say C, has w levels C_1, C_2, ..., C_w. Thus, the v (= stw) denotes the number of all treatment combinations in the experiment.

In the SBP design we assume that experimental material can be divided into b blocks. Every block forms a row-column design with k_1 rows and k_2 columns. Then, each intersection plot (called a whole plot) can be divided into k_3 subplots. Here the rows correspond to the levels of the factor A, termed as row treatments, the columns correspond to the levels of the factor B, called column treatments, and the subplots are to accommodate the levels of the factor C, termed as subplot treatments.

In the SPSB design every block also forms a row-column design with k_1 rows and k_2 columns of the first order, called I-columns for short. Then each

I-column has to be split into k_3 columns of the second order (called II-columns). In this case, the rows also correspond to the levels of the factor *A*, termed as row treatments, the I-columns correspond to the levels of the factor *B*, termed as I-column treatments, and the II-columns are to accommodate the levels of the factor *C*, termed as II-column treatments. Therefore, n $(=bk_1k_2k_3)$ denotes a number of the subplots, required in both designs.

3. Linear models

Consider randomization models of observations, the forms and properties of which are strictly connected with the performed randomization processes in experiments. The randomization scheme of the SBP design consists of the four randomization steps performed independently, namely by randomly permuting blocks, rows, columns and subplots. Next the randomization scheme used in the SPSB design also consists of the four randomization steps performed independently, that is by randomly permuting blocks, rows, I-columns and II-columns. The resulting mixed models of the SBP and SPSB designs can be written in the following form:

$$\mathrm{E}(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau}, \ \mathrm{Cov}(\mathbf{y}) = \mathbf{V}(\gamma), \tag{1}$$

where $\boldsymbol{\Delta}'$ is a known design matrix for v treatment combinations, and $\boldsymbol{\tau}$ $(v \times 1)$ is the vector of fixed treatment combination effects. According to the orthogonal block structure of the designs, the dispersion matrix $\mathbf{V}(\gamma)$ can be expressed by $\mathbf{V}(\gamma) = \sum_{f=0}^{m} \gamma_f \mathbf{P}_f$ ($m = 5$ for the SBP design or $m = 6$ for the SPSB design), where γ_f (≥0) are unknown stratum variances and $\{\mathbf{P}_f\}$ constitute a set of known pair-wise orthogonal projectors adding up to the identity matrix (e.g. Ambroży and Mejza, 2002, 2004).

In the SBP design

$$\gamma_0 = \sigma_e^2, \ \gamma_1 = k_1k_2k_3\sigma_1^2 + \sigma_e^2, \ \gamma_2 = k_2k_3\sigma_2^2 + \sigma_e^2,$$

$$\gamma_3 = k_1k_3\sigma_3^2 + \sigma_e^2, \ \gamma_4 = k_3\sigma_4^2 + \sigma_e^2, \ \gamma_5 = \sigma_5^2 + \sigma_e^2 \tag{2}$$

where σ_f^2 $(f = 1, 2, ..., 5)$ denote, respectively, variance components related to the blocks, the rows, the columns, the whole plots, the subplots and σ_e^2 denotes a variance of technical errors.

In the SPSB design

$$\begin{aligned}
&\gamma_0 = \sigma_e^2,\ \gamma_1 = k_1 k_2 k_3 \sigma_1^2 + \sigma_e^2,\ \ \gamma_2 = k_2 k_3 \sigma_2^2 + \sigma_e^2,\\
&\gamma_3 = k_1 k_3 \sigma_3^2 + \sigma_e^2,\ \gamma_4 = k_1 \sigma_4^2 + \sigma_e^2,\ \gamma_5 = k_3\sigma_5^2 + \sigma_e^2,\\
&\gamma_6 = \sigma_6^2 + \sigma_e^2,
\end{aligned} \tag{3}$$

where σ_f^2 $(f = 1, 2, ..., 6)$ denote, respectively, variance components related to the blocks, the rows, the I-columns, the II-columns, the whole plots, the subplots and σ_e^2 - how above.

The models (1) can be analyzed in accordance with the methods developed for multistratum experiments. The range space $\Re\{\mathbf{P}_f\}$ of $\mathbf{P}_f$, $f = 0,1,..,m$, is termed the *f*-th stratum of the model (e.g. Houtman and Speed, 1983).

In the SBP design there are five main strata connected with an estimation of the contrasts: an inter-block stratum (1), an inter-row (within the blocks) stratum (2), an inter-column (within the blocks) stratum (3), an inter-whole plot (within the blocks) stratum (4) and an inter-subplot (within the whole plots) stratum (5).

In the SPSB design there are six main strata: an inter-block stratum (1), an inter-row (within the blocks) stratum (2), an inter-I-column (within the blocks) stratum (3), an inter-II-column (within the I-columns) stratum (4), an inter-whole plot (within the blocks) stratum (5) and an inter-subplot (within the whole plots) stratum (6). Additionally, in both models, there is the so-called zero stratum (0) connected with mean estimation only.

The statistical analysis of such models is connected with algebraic properties of stratum information matrices for the treatment combinations, $\mathbf{A}_f$ (e.g. Ambroży and Mejza, 2002, 2004). The eigenvalues of them, say ε_{fh}, are identified as stratum efficiency factors of the design with respect to a set of orthogonal contrasts defined by the eigenvectors of these matrices. In complete designs (i.e. when k_1=*s*, k_2=*t*, k_3=*w*) certain groups of the contrasts are estimated in one stratum only, appropriate for them, with full efficiency, then ε_{fh} =1. We can express it using an abbreviation as follows.

Let $M_f\{q, 1\}$ denote the property that q contrasts among levels of factor M (or interaction contrasts) are estimated with full efficiency ($\varepsilon_{fh} = 1$) in the f-th stratum. In other words, we say that the design is $M_f\{q, 1\}$ - orthogonal (e.g. Mejza and Ambroży, 2003, Ambroży and Mejza, 2004).

Corollary 1. *The complete SBP design is*: $A_2\{s–1, 1\}$ - *orthogonal*, $B_3\{t–1, 1\}$ - *orthogonal*, $C_5\{w–1, 1\}$ - *orthogonal*, $(A\times B)_4\{(s–1)(t–1), 1\}$ - *orthogonal*, $(A\times C)_5\{(s–1)(w–1), 1\}$ - *orthogonal*, $(B\times C)_5\{(t–1)(w–1), 1\}$ - *orthogonal and* $(A\times B\times C)_5\{(s–1)(t–1)(w–1), 1\}$ - *orthogonal*.

Corollary 2. *The complete SPSB design is*: $A_2\{s–1, 1\}$ - *orthogonal*, $B_3\{t–1, 1\}$ - *orthogonal*, $C_4\{w–1, 1\}$ - *orthogonal*, $(A\times B)_5\{(s–1)(t–1), 1\}$ - *orthogonal*, $(A\times C)_6\{(s–1)(w–1), 1\}$ - *orthogonal*, $(B\times C)_4\{(t–1)(w–1), 1\}$ - *orthogonal and* $(A\times B\times C)_6\{(s–1)(t–1)(w–1), 1\}$ - *orthogonal*.

4. Efficiency comparison of SPSB designs versus SBP designs (complete cases)

If we plan to carry out an experiment in one of these designs, it is necessary to take into account a research problem, an available experimental material and first of all technical reasons. It is also necessary to think, when we lose and when we benefit from estimation of the treatment parameters using the SBP design, relatively SPSB design, with the same number of experimental units. To examine the effectiveness of both considered designs in the context of point estimation we can use *the relative efficiency* introduced by Yates (1935) as comparison of the accuracy of the estimation (e.g. Hinkelmann and Kemptorne, 1994, Hering and Wang, 1998, Wang, 2002, Wang and Hering, 2005, Shieh and Jan, 2004).

Definition 1. Let Γ_1 and Γ_2 denote any experimental designs, then relative efficiency of these designs (Γ_1 versus Γ_2) is defined as:

$$\mathrm{RE}(\Gamma_1/\Gamma_2)=\frac{\mathrm{Var}\Gamma_2}{\mathrm{Var}\Gamma_1} \tag{4}$$

where Var Γ_1 and Var Γ_2 denote variances of the same contrast in both designs.

The relative efficiency as defined in (4) depends on the true stratum variances of both designs, which usually are unknown. Moreover, the stratum variances are functions of variance components defined during the randomization processes. Relations among them allow comparing efficiencies of the SSP and SBP designs. Usually the same relations occur among estimates of the stratum variances (except for sampling errors). Comparing an efficiency of the SPSB design in a relation to an efficiency of the SBP design for an estimation of contrasts, in some cases we can use the measure defined in (4) but in others we should take into account the estimation of the RE, called empirical relative efficiency, which we shall denote by ERE.

For the SBP design it could be expected that

$$\sigma_1^2>\sigma_2^2>\sigma_4^2>\sigma_5^2>\sigma_e^2 \quad \text{and} \quad \sigma_1^2>\sigma_3^2>\sigma_4^2>\sigma_5^2>\sigma_e^2,$$

which imply the inequalities (see (2)):

$$\gamma_1>\gamma_2>\gamma_4>\gamma_5 \quad \text{and} \quad \gamma_1>\gamma_3>\gamma_4>\gamma_5. \tag{5}$$

For the SPSB design it could be expected that:

$$\sigma_1^2>\sigma_2^2>\sigma_5^2>\sigma_6^2>\sigma_e^2 \quad \text{and} \quad \sigma_1^2>\sigma_3^2>\sigma_4^2>\sigma_5^2>\sigma_6^2>\sigma_e^2,$$

which imply the inequalities (see (3)):

$$\gamma_1>\gamma_2>\gamma_5>\gamma_6 \quad \text{and} \quad \gamma_1>\gamma_3>\gamma_4>\gamma_5>\gamma_6. \tag{6}$$

Therefore, it can be assumed (except for sampling errors) that estimates of variance components $\hat{\gamma}_f$ in both designs also satisfy the inequalities (5) and (6). Next to define the relations between the appropriate errors, located in both designs, “*blind*” *experiments* (for example field was sowed one variety) are considered. Then ANOVAs for the SBP design and the SPSB design are given in the Table 1.

We can see from Table 1, that

$$(b-1)\,\hat{\gamma}_1^{SBP} + b(s-1)\,\hat{\gamma}_2^{SBP} + b(t-1)\,\hat{\gamma}_3^{SBP} + b(s-1)(t-1)\,\hat{\gamma}_4^{SBP} + bst(w-1)\,\hat{\gamma}_5^{SBP} = (b-1)\,\hat{\gamma}_1^{SPSB} + b(s-1)\,\hat{\gamma}_2^{SPSB} + b(t-1)\,\hat{\gamma}_3^{SPSB} + bt(w-1)\,\hat{\gamma}_4^{SPSB} + b(s-1)(t-1)\,\hat{\gamma}_5^{SPSB} + bt(s-1)(w-1)\,\hat{\gamma}_6^{SPSB} \quad (7)$$

Table 1. ANOVA of a "blind" experiment

a) the SBP design

Sources	DF	Mean squares ($\hat{\gamma}_f = MSE_f$)
(1) Blocks	$b-1$	$\hat{\gamma}_1 = MSE_1$
(2) Rows	$b(s-1)$	$\hat{\gamma}_2 = MSE_2$
(3) Columns	$b(t-1)$	$\hat{\gamma}_3 = MSE_3$
(4) Whole plots	$b(s-1)(t-1)$	$\hat{\gamma}_4 = MSE_4$
(5) Subplots	$bst(w-1)$	$\hat{\gamma}_5 = MSE_5$
Total	$bstw-1$	

b) the SPSB design

Sources	DF	Mean squares ($\hat{\gamma}_f = MSE_f$)
(1) Blocks	$b-1$	$\hat{\gamma}_1 = MSE_1$
(2) Rows	$b(s-1)$	$\hat{\gamma}_2 = MSE_2$
(3) I-columns	$b(t-1)$	$\hat{\gamma}_3 = MSE_3$
(4) II-columns	$bt(w-1)$	$\hat{\gamma}_4 = MSE_4$
(5) Whole plots	$b(s-1)(t-1)$	$\hat{\gamma}_5 = MSE_5$
(6) Subplots	$bt(s-1)(w-1)$	$\hat{\gamma}_6 = MSE_6$
Total	$bstw-1$	

From (2) and (3), it can be expected that the appropriate stratum variances, thus also their estimates, in both designs will be identical (to some extent)

$$\hat{\gamma}_1^{\mathrm{SBP}} = \hat{\gamma}_1^{\mathrm{SPSB}},\ \hat{\gamma}_2^{\mathrm{SBP}} = \hat{\gamma}_2^{\mathrm{SPSB}},\ \hat{\gamma}_3^{\mathrm{SBP}} = \hat{\gamma}_3^{\mathrm{SPSB}},\ \hat{\gamma}_4^{\mathrm{SBP}} = \hat{\gamma}_5^{\mathrm{SPSB}}. \quad (8)$$

It was one should underline, that $\hat{\gamma}_1^{\mathrm{SBP}} = \hat{\gamma}_1^{\mathrm{SPSB}} = 0$, if both designs are complete. Thus, (7) simplifies to

$$bst(w-1)\ \hat{\gamma}_5^{\mathrm{SBP}} = bt(w-1)\ \hat{\gamma}_4^{\mathrm{SPSB}} + bt(s-1)(w-1)\ \hat{\gamma}_6^{\mathrm{SPSB}},$$

and hence

$$\hat{\gamma}_5^{SBP} = \frac{bt(w-1)\hat{\gamma}_4^{SPSB} + bt(s-1)(w-1)\hat{\gamma}_6^{SPSB}}{bst(w-1)}. \quad (9)$$

The inequality (6) implies that $\hat{\gamma}_4^{\mathrm{SPSB}} > \hat{\gamma}_6^{\mathrm{SPSB}}$. It means that the last error (subplot error) in the SBP design is a weighted average between the two errors in the SPSB design, i.e.

$$\hat{\gamma}_6^{\mathrm{SPSB}} < \hat{\gamma}_5^{\mathrm{SBP}} < \hat{\gamma}_4^{\mathrm{SPSB}}. \quad (10)$$

The relations (5) - (10) were applied to examine the empirical relative efficiency of the complete SPSB and SBP designs for an estimation of the orthogonal contrasts connected with main and interaction effects of the factors. In this connection, let $K = \{h: h = 1, 2, ..., v-1\}$ as well as let K_A, K_B, K_C, $K_{A\times B}$, $K_{A\times C}$, $K_{B\times C}$, $K_{A\times B\times C}$ denote sets of numbers of orthogonal contrasts connected with main effects and different types of interaction effects of the factors *A*, *B*, *C* and $K_A \cup K_B \cup K_C \cup K_{A\times B} \cup K_{A\times C} \cup K_{B\times C} \cup K_{A\times B\times C} = K$

First let us consider the relative efficiency of both designs for estimation of the contrasts connected with main effects and interaction effects of factors *A* and *B*. Let $\boldsymbol{c}_h'\boldsymbol{\tau}$, where $h \in K_A$ or $h \in K_B$ or $h \in K_{A\times B}$ denote this contrast. From Corollaries 1 and 2 it follows that these contrasts are estimable, respectively, in the inter-row stratum (2), in the inter-column stratum (in the inter-I-column stratum if a SPSB design is used) - (3) and in the inter-whole plot stratum (in the SBP design it is the fourth stratum and in the SPSB design - the fifth one) in both complete designs. From (8) we have, that suitable stratum variances are the same for above-mentioned sets of contrasts. So, in this case it is meaningless if the variances are known or not. We have then

Corollary 3. A measure of the efficiency of a complete SPSB design relative to a complete SBP design (with the same number of experimental units) for each h-th contrast, where $h \in K_A \cup K_B \cup K_{A\times B}$ is as follows

$$\mathrm{RE}_h^A\,(\mathrm{SPSB/SBP}) = \mathrm{RE}_h^B\,(\mathrm{SPSB/SBP}) = \mathrm{RE}_h^{A\times B}\,(\mathrm{SPSB/SBP}) = 1.$$

The next corollary defines the effectiveness of the SPSB and SBP designs in the estimation of the orthogonal contrasts among main effects of the factor C and the interaction contrasts of $B\times C$ type. From Corollaries 1 and 2 follows that in a complete SPSB design these comparisons are estimable in the inter-II-column stratum (4), however in a SBP design - in the inter-subplot stratum (5). Let $\boldsymbol{c}_h'\boldsymbol{\tau}$, where $h \in K_C$ or $h \in K_{B\times C}$ denote this contrast. Then, from relation (10) it follows that

Corollary 4. A measure of the efficiency of a complete SPSB design relative to a complete SBP design (with the same number of experimental units) for each h-th contrast, where $\mathrm{h} \in \mathrm{K_C} \cup \mathrm{K_{B\times C}}$ is following

$$\mathrm{ERE}_h^C\,(\mathrm{SPSB\,/\,SBP}) = \mathrm{ERE}_h^{B\times C}\,(\mathrm{SPSB\,/\,SBP}) =$$

$$= \frac{\widehat{\mathrm{Var}}^{\mathrm{SBP}}[(\widehat{\mathbf{c}_h'\boldsymbol{\tau}})_5]}{\widehat{\mathrm{Var}}^{\mathrm{SPSB}}[(\widehat{\mathbf{c}_h'\boldsymbol{\tau}})_4]} = \frac{\hat{\gamma}_5^{\mathrm{SBP}}}{\hat{\gamma}_4^{\mathrm{SPSB}}} < 1.$$

The last conclusion concerns the relative efficiency of the considered designs for the estimation of the interaction contrasts of types $A\times C$ and $A\times B\times C$ as well. From Corollaries 1 and 2 we know that full information about these contrasts is in the inter-subplots, i.e. respectively in the sixth and fifth stratum of SPSB and SBP designs. Let $\boldsymbol{c}_h'\boldsymbol{\tau}$, where $h \in K_{A\times C}$ or $h \in K_{A\times B\times C}$ denote interaction contrast. Consequently, from the relation (10) it follows that

Corollary 5. A measure of the efficiency of a complete SPSB design relative to a complete SBP design (with the same number of experimental units) for each h-th contrast, where $\mathrm{h} \in \mathrm{K_{A\times C}} \cup \mathrm{K_{A\times B\times C}}$ is

$$\mathrm{ERE}_h^{A\times C}(\mathrm{SPSB}\,/\,\mathrm{SBP}) = \mathrm{ERE}_h^{A\times B\times C}(\mathrm{SPSB}\,/\,\mathrm{SBP}) =$$

$$= \frac{\widehat{\mathrm{Var}}^{\mathrm{SBP}}[(\widehat{\mathbf{c}_h'\boldsymbol{\tau}})_5]}{\widehat{\mathrm{Var}}^{\mathrm{SPSB}}[(\widehat{\mathbf{c}_h'\boldsymbol{\tau}})_6]} = \frac{\hat{\gamma}_5^{\mathrm{SBP}}}{\hat{\gamma}_6^{\mathrm{SPSB}}} > 1.$$

5. Conclusions

1) The efficiency of a complete SPSB design relative to a complete SBP design with the same number of experimental units (= n) was investigated. We obtained as follows:
   a) Both compared designs are always equally effective for estimation of the contrasts among main effects of the factors A and B and the interaction effects of the $A\times B$ type, i.e. a precision of their estimation is the same in both designs (Corollary 3).
   b) For the estimation of the contrasts connected with main effects of factor C and its interaction effects connected with factor B the SBP design is more effective than the SPSB design. Hence, the accuracy of comparisons estimation of these treatment parameters is usually bigger in the SBP design than in the SPSB design (Corollary 4).
   c) Next, the SPSB design is more effective than the SBP design in the estimation of the interaction contrasts of types $A\times C$ and $A\times B\times C$. The accuracy of comparisons estimation of these treatment parameters is usually bigger in the SPSB design (Corollary 5).

2) The above-mentioned conclusions cannot be generally extended over all (planned) incomplete designs, i.e. when $k_1 < s$ or/and $k_2 < t$ or/and $k_3 < w$. It is known that the variance of a contrast estimator in a general model is a function of variances of BLUEs of the contrast in strata (e.g. Searle, 1971). The form of this function depends on unknown stratum variances, γ_f and stratum efficiency factors ε_{fh}. Although there are certain relations among them (see (5) and (6)) as well as stratum efficiency factors sum up to one, $\sum_f \varepsilon_{fh} = 1$, there is no connection among variances of the stratum estimators. Therefore, we can

compare efficiencies of the incomplete designs generated by the same method, in suitable strata only, in which the given contrast is estimable. If the generating designs for one or more factors are efficiency balanced (in particular BIB designs), it could be expected that main (though not only) sources of information about the contrasts will be identified with the strata as in the complete SPSB and SBP designs. From the fact that the generating designs at the factors are the same respectively in both mixed designs, we obtain that corresponding stratum efficiency factors are identical in the mixed designs. Conclusions then concerning the relative efficiency of the incomplete SPSB design versus the incomplete SBP design in the strata with the largest number (suitable for an estimable contrast) agree with the conclusions in (1a)–(1c).

6. A practical example

To illustrate the theory presented in the paper, consider $2 \times 5 \times 2$ experiment designed to test the effects of two (s = 2) levels of nitrogen fertilization (kg/ha) A_1 - 90, A_2 - 150 and two (w = 2) chemical preparation - growth regulator (kg/ha) C_1 - 0 , C_2 - 2 on the grain yields of five (t = 5) wheat varieties B_1 - Grana, B_2 - Dana, B_3 - Eka Nowa, B_4 - Kaukaz, B_5 - Mironowskaja 808. Original experiment was carried out in Słupia Wielka (Poland) in the complete SPSB design, in three blocks (replications). Then for a comparing, the same data (table 4) were analyzed under mixed linear model as data from the complete SBP design (both analyzes were performed with the help of *STATISTICA* program). The tables 2 and 3 present the ANOVA for the complete SBP and SPSB designs, respectively.

In the Table 2, Error (f) , f = 2, 3, 4, 5 denotes error in the f-th stratum, i.e. the row-stratum, the column-stratum, the whole plot-stratum and the subplot-stratum. In the Table 3, Error (f) , f = 2, 3, 4, 5, 6 denotes error in the f-th stratum, i.e. the row-stratum, the I-column stratum, the II-column stratum, the whole plot-stratum and the subplot-stratum. In both designs, Error (1) is equal to 0 because they are complete.

Table 2. ANOVA for the complete SBP design

Source	Effect	SS	DF	MS	F	P
Blocks	Random	165.9523	2	82.9762		
A	Fixed	302.8507	1	302.8507	102.45	0.0096
Error (2)	Random	5.9123	2	2.9562		
B	Fixed	493.6123	4	123.4031	47.09	0.0000
Error (3)	Random	20.9627	8	2.6203		
A x *B*	Fixed	28.7510	4	7.1878	3.13	0.0794
Error (4)	Random	18.3660	8	2.2958		
C	Fixed	216.6000	1	216.6000	197.87	0.0000
A x *C*	Fixed	2.0907	1	2.0907	1.91	0.1822
B x *C*	Fixed	58.1717	4	14.5429	13.29	0.0000
A x *B* x *C*	Fixed	13.6743	4	3.4186	3.12	0.0378
Error (5)		21.8933	20	1.0947		

Table 3. ANOVA for the complete SPSB design

Source	Effect	SS	DF	MS	F	P
Blocks	Random	165.9523	2	82.9762		
A	Fixed	302.8507	1	302.8507	102.45	0.0096
Error (2)	Random	5.9123	2	2.9562		
B	Fixed	493.6123	4	123.4031	47.09	0.0000
Error (3)	Random	20.9627	8	2.6203		
C	Fixed	216.6000	1	216.6000	180.21	0.0000
B x *C*	Fixed	58.1717	4	14.5429	12.10	0.0008
Error (4)	Random	12.0186	10	1.2019		
A x *B*	Fixed	28.7510	4	7.1878	3.13	0.0794
Error (5)	Random	18.3660	8	2.2958		
A x *C*	Fixed	2.0907	1	2.0907	2.12	0.1763
A x *B* x *C*	Fixed	13.6743	4	3.4186	3.46	0.0506
Error (6)		9.8750	10	0.9875		

It can be noticed both from tables 2 and 3 and from Corollary 3 as well that for each h-th ($h \in K_A$) contrast connected with main effects of the nitrogen fertilization (A), for each h-th ($h \in K_B$) contrast connected with main effects of the wheat varieties (B) and for each h-th ($h \in K_{A\times B}$) interaction contrast of type $A \times B$ both considered designs are always equally effective.

Table 4. The data set from the $2 \times 5 \times 2$ experiment

Blocks	*A*	*B*	*C*	Grain yield
1	90	Grana	0	34.2
1	90	Grana	2	39.1
1	90	Dana	0	30.8
1	90	Dana	2	33.8
1	90	Eka Nowa	0	31.0
1	90	Eka Nowa	2	33.5
1	90	Kaukaz	0	32.3
1	90	Kaukaz	2	33.5
1	90	Mironowskaja 808	0	33.5
1	90	Mironowskaja 808	2	36.0
1	150	Grana	0	33.2
1	150	Grana	2	31.0
1	150	Dana	0	29.8
1	150	Dana	2	31.5
1	150	Eka Nowa	0	29.9
1	150	Eka Nowa	2	30.8
1	150	Kaukaz	0	28.8
1	150	Kaukaz	2	35.0
1	150	Mironowskaja 808	0	28.5
1	150	Mironowskaja 808	2	30.0
2	90	Grana	0	39.9
2	90	Grana	2	40.0
2	90	Dana	0	42.0
2	90	Dana	2	43.9
2	90	Eka Nowa	0	39.0
2	90	Eka Nowa	2	43.5
2	90	Kaukaz	0	43.1
2	90	Kaukaz	2	46.2
2	90	Mironowskaja 808	0	34.4

2	90	Mironowskaja 808	2	36.1
2	150	Grana	0	42.8
2	150	Grana	2	41.2
2	150	Dana	0	32.1
2	150	Dana	2	35.9
2	150	Eka Nowa	0	34.0
2	150	Eka Nowa	2	36.9
2	150	Kaukaz	0	37.6
2	150	Kaukaz	2	39.5
2	150	Mironowskaja 808	0	34.0
2	150	Mironowskaja 808	2	37.0
3	90	Grana	0	44.0
3	90	Grana	2	41.2
3	90	Dana	0	39.8
3	90	Dana	2	43.8
3	90	Eka Nowa	0	39.0
3	90	Eka Nowa	2	43.8
3	90	Kaukaz	0	42.2
3	90	Kaukaz	2	46.0
3	90	Mironowskaja 808	0	36.3
3	90	Mironowskaja 808	2	42.5
3	150	Grana	0	39.8
3	150	Grana	2	37.2
3	150	Dana	0	35.6
3	150	Dana	2	39.7
3	150	Eka Nowa	0	32.8
3	150	Eka Nowa	2	35.8
3	150	Kaukaz	0	35.8
3	150	Kaukaz	2	39.0
3	150	Mironowskaja 808	0	34.0
3	150	Mironowskaja 808	2	37.1

Then, for each h-th ($h \in K_C \cup K_{B\times C}$) contrast connected with main effects of the growth regulator or an interaction contrast of type $B \times C$

$$\mathrm{ERE}_h^{C}\,(\mathrm{SPSB}\,/\,\mathrm{SBP}) = \mathrm{ERE}_h^{B\times C}\,(\mathrm{SPSB}\,/\,\mathrm{SBP}) = \frac{MSE_5^{\mathrm{SBP}}}{MSE_4^{\mathrm{SPSB}}} = \frac{1.0947}{1.2019} < 1\,.$$

Lastly, for each h–th ($h \in K_{A\times C} \cup K_{A\times B\times C}$) interaction contrast of type $A \times C$ or $A \times B \times C$

$$\mathrm{ERE}_h^{A\times C}\,(\mathrm{SPSB}\,/\,\mathrm{SBP}) = \mathrm{ERE}_h^{A\times B\times C}\,(\mathrm{SPSB}\,/\,\mathrm{SBP}) = \frac{MSE_5^{\mathrm{SBP}}}{MSE_6^{\mathrm{SPSB}}} = \frac{1.0947}{0.9875} > 1\,.$$

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